

Clifford-Wolf homogeneous left invariant (α, β) -metrics on compact Lie groups*

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Abstract

Let (M, F) be a connected Finsler space. An isometry of (M, F) is called a Clifford-Wolf translation (or simply CW-translation) if it moves each point the same distance. The space (M, F) is called Clifford-Wolf homogeneous (CW-homogeneous) if for any two points $x_1, x_2 \in M$, there exists a CW-translation σ such that $\sigma(x_1) = x_2$. In this paper, we study CW-homogeneous left invariant (α, β) -metrics on compact Lie groups. We show that such a metric must be of the Randers type. This gives a complete classification of left invariant (α, β) -metrics on compact simple Lie groups which are Clifford-homogeneous.

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1 Introduction

The goal of this paper is to study CW-homogeneous left invariant (α, β) -metrics on compact Lie groups. Recall that an isometry σ of a metric space (X, d) is called a CW-translation if the function $d(x, \sigma(x))$, $x \in X$ is a constant. The metric space is called CW-homogeneous if given any two points $x_1, x_2 \in M$, there exists a CW-translation σ such that $\sigma(x_1) = x_2$; see [BP99]. Although generally a Finsler metric is not reversible, the above definitions can be adapted to Finsler spaces by a word by word restatement.

The study of CW-translations has important merits in the investigations of space forms in Riemannian geometry; see Wolf's book [WO10] for an excellent survey. The related results has motivated a lot of mathematical activities; see for example [WO62, WO64, FR63, OZ69, OZ74, DMW86] for the determination of CW-translations

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of explicit Riemannian manifolds; see also [HE74, AW76] for the applications of the related results to the study of homogeneous Riemannian manifolds of negative (non-positive) curvature.

Recently, Berestovskii and Nikonorov studied the local one-parameter groups of CW-translations of general Riemannian manifolds and established a correspondence between local one-parameter groups of CW-translations and Killing vector fields of constant length; see [BN08-1, BN08-2, BN09]. The above research lead to a classification of connected simply connected CW-homogeneous Riemannian manifolds. The list consists of Euclidean spaces, odd-dimensional spheres with constant curvature, connected simply compact connected simple Lie groups with bi-invariant Riemannian metrics and direct products of the above manifolds. Notice that there is also a local version of the related notion of CW-homogeneous Riemannian manifolds; see the above cited references.

More recently, we initiated the study of CW-translations of Finsler spaces; see [DM1] and [DM2]. The relationship between local one-parameter group of CW-translations and Killing vector fields of constant length has been generalized to the Finslerian case. We have also given a complete classification of CW-homogeneous left invariant Randers metrics on compact simple Lie groups ([DM3]). In this paper we consider a more generalized class of Finsler metrics, (α, β) -metrics on compact Lie groups. The main result is the following

Theorem 1.1 *Suppose $F = \alpha\phi(\frac{\beta}{\alpha})$ is a CW-homogeneous left invariant (α, β) -metric on a compact connected simple Lie group. If F is regular, i.e., if $\phi'(0) \neq 0$, then F must be a Randers metric.*

In view of the main result of our previous paper ([DM3]), this theorem gives a complete classification of CW-homogeneous left invariant (α, β) -metrics on compact simple Lie groups. It would be an interesting problem to classify all the CW-homogeneous Finsler spaces.

In Section 2, we present some known results on related topics. Section 3 is devoted to the study of general geometric properties of (α, β) -metrics. Results in this section has the general merits in Finsler geometry. In Section 4, we study Killing vector fields of constant length of left invariant (α, β) -metrics on compact Lie groups. Finally, in Section 5, we prove the main result of this paper.

2 Preliminaries

Let M be a connected smooth manifold. A Finsler metric on M is a continuous function: $F : TM \rightarrow [0, \infty)$ satisfying the following properties: for any local coordinates (x, y) for TM , where $x = (x^i)$ gives a local coordinates on M , and $y = y^j \frac{\partial}{\partial x^j} \in TM_x$:

- (1) (Positivity) $F(x, y)$ is a smooth positive function on the slit tangent bundle $TM \setminus 0$.
- (2) (Positive homogeneity) $F(x, \lambda y) = \lambda F(x, y)$ for any $\lambda > 0$.
- (3) (Strong convexity) the Hessian matrix $g_{ij}(y) = \frac{1}{2}[F^2]_{y^i y^j}$ is positive definite on $TM \setminus 0$.

The pair (M, F) is called a Finsler space (or a Finsler manifold).

The most important class of Finsler metrics are the Randers metrics. A Finsler metric of the form $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form, is called a Randers metric. Randers metrics were introduced by G. Randers in 1941, in his study of general relativity ([RA41]). Using the positivity of F , one easily deduces that the α -norm of β must be everywhere less than 1.

There is a more generalized class of Finsler metrics which have been studied extensively in the literature. Let α be a Riemannian metric on a manifold M and β be a smooth 1-form. A Finsler metric of the form $F = \alpha\phi(\frac{\beta}{\alpha})$, where ϕ is a smooth real function on \mathbb{R} , is called an (α, β) -metric. We now recall the conditions for such a metric to be positive and strong convex. Denote $\varepsilon_0 = \sup_{y \in TM \setminus 0} \frac{\beta(y)}{\alpha(y)}$. If ε_0 can be attained at certain y_0 , then ϕ is required to be a positive smooth function on $I = [-\varepsilon_0, \varepsilon_0]$ ($I = \mathbb{R}$ when $\varepsilon_0 = \infty$), satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad (2.1)$$

for all b and s such that $|s| \leq |b| \leq \varepsilon_0$. If ε_0 can not be attained at any point, then ϕ is required to be a positive smooth function on $(-\varepsilon_0, \varepsilon_0)$, and (2.1) is required for all b and s with $|s| \leq |b| < \varepsilon_0$. Notice that neither α nor β in this term is relevant to the Riemannian metric or the 1-form used to define the metric. The inequality (2.1) is a necessary and sufficient condition derived from the positivity and strong convexity of the Finsler metric. When β is identically 0 (and then (2.1) reduces to $\phi(0) > 0$), or ϕ is a constant function, F is Riemannian. If ϕ is a linear function, then F is a Randers metric.

On a Finsler space (M, F) one can define the arc length of a piecewise smooth path. The supremum among the arc lengths of all piecewise smooth paths from one point to another is the distance function, denoted as $d(\cdot, \cdot)$. This distance function does not satisfy the reversibility $d(x, x') \equiv d(x', x)$, unless F is reversible, i.e., $F(x, y) = F(x, -y)$ for any $x \in M, y \in T_x(M)$. An isometry φ of the Finsler space (M, F) can be defined as a diffeomorphism of M with $\varphi^*F = F$, or equivalently a homeomorphism of M with $d(x, x') = d(\varphi(x), \varphi(x'))$ for all $x, x' \in M$. It has been proven by Deng and Hou that the group $I(M, F)$ of all isometries of (M, F) , with the open-compact topology, is a Lie group. If $I(M, F)$ acts transitively on M , then (M, F) is called a homogeneous Finsler space, and the manifold M can be denoted as a quotient G/H , in which G is any closed subgroup of $I(M, F)$ which acts transitively on M and H is the isotropy subgroup of a point $x \in M$ in G . In general, there may be more than one way to present M as G/H . Because we will always assume M to be connected, the subgroup G can also be chosen from closed connected subgroups of $I_0(M, F) \subset I(M, F)$.

The notions of CW-translations and CW-homogeneity of Finsler spaces can be defined in the same way as for metric spaces. For the completeness of the article we briefly recall the definitions here.

Definition 2.1 *An isometry ρ of (M, F) is called a Clifford-Wolf translation (CW-translation) if $d(x, \rho(x))$ is a constant function for $x \in M$.*

Definition 2.2 *A Finsler space (M, F) is called Clifford-Wolf homogeneous (CW-homogeneous) if for any pair $x, x' \in M$, there is a CW-translation ρ which maps x*

to x' . It is called *restrictively CW-homogeneous*, if for any x , there is a neighborhood U of x , such that for any pair x and x' in U , there is a CW-translation ρ of (M, F) , such that $\rho(x) = x'$.

The main tool to study CW-translations and CW-homogeneity in Finsler geometry is a natural interrelation between Killing vector fields of constant length and local one-parameter semigroups of CW-translations. We now recall the main results.

Theorem 2.3 *Suppose a complete Finsler manifold (M, F) has positive injective radius. If X is a Killing field on (M, F) of constant length and φ_t is the flow generated by X , then φ_t is a Clifford Wolf translation for any sufficiently small $t > 0$.*

Theorem 2.4 *Let (M, F) be a compact Finsler space. Then there is a $\delta > 0$, such that any Clifford-Wolf translation ρ with $d(x, \rho(x)) < \delta$ is generated by a Killing vector field of constant length.*

Notice that Theorem 2.4 is still correct if we replace the compactness of M by the homogeneity of (M, F) .

Based on these interrelation theorems, we have an equivalent description of restrictive CW-homogeneity.

Theorem 2.5 *Let (M, F) be a connected homogeneous Finsler space. Then it is restrictively CW-homogeneous if and only if the Killing fields of constant length can exhaust all tangent directions, i.e., any geodesic starting from any point is the flow curve of a Killing field of constant length.*

3 Regularity and homogeneity of (α, β) -spaces

In this section we study some geometric properties of (α, β) -metrics. The results will be of general merits in Finsler geometry. The main focus is on the regularity and homogeneity of (α, β) -metrics. The homogeneity of a Finsler space (M, F) can help us setup a correspondence between Finsler metrics and Minkowski norms on a fixed tangent space. Furthermore, for some particular type of Finsler metrics, such as Riemannian metrics or Randers metrics, the metric can be uniquely determined by some algebraic data which is invariant under the action of the isotropy group. For example, suppose $F = \alpha + \beta$ is a homogeneous Randers metric on a coset space $M = G/H$. Denote $\text{Lie}(G) = \mathfrak{g}$, $\text{Lie}(H) = \mathfrak{h}$ and $TM_x \cong \mathfrak{m} = \mathfrak{g}/\mathfrak{h}$. Then β is determined by a vector in \mathfrak{m}^* and α is determined by an inner product on \mathfrak{m} , both being $\text{Ad}(H)$ -invariant.

However, for homogeneous (α, β) -metrics the situation is somehow more complicated. This is mainly due to the fact that the presentation $F = \alpha\phi(\frac{\beta}{\alpha})$ of an (α, β) -metric, i.e., the triple α , β and ϕ may not be uniquely determined by F . To tackle this exotic situation, we define the notion of regular (α, β) -metrics.

Definition 3.1 *An (α, β) -metric $F = \alpha\phi(\frac{\beta}{\alpha})$ is called regular, if $\phi'(0) \neq 0$.*

Given a fixed ϕ with $\phi'(0) \neq 0$, the regular (α, β) -Minkowski norm of the form $F = \alpha\phi(\frac{\beta}{\alpha})$ on \mathbb{R}^n , with $n > 2$, is uniquely determined by α and β .

Lemma 3.2 Suppose the function ϕ satisfies $\phi'(0) \neq 0$ and $\alpha_1\phi(\frac{\beta_1}{\alpha_1}) = \alpha_2\phi(\frac{\beta_2}{\alpha_2})$ defines the same Minkowski space (\mathbb{R}^n, F) , where $n > 2$. Then $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

Proof. We first prove the lemma under the assumption that β_1 and β_2 are linearly dependent. The requirement on the dimension is not needed in this step.

The case $\beta_1 = \beta_2 = 0 \in \mathbb{R}^{n*}$ is trivial. So we may assume that $\beta_1 \neq 0$ and $\beta_2 = \lambda\beta_1$. By choosing suitable linear coordinates, and scalar changes, we can assume that $\alpha_1^2 = y_1^2 + \dots + y_{n-1}^2$, where y_1, \dots, y_{n-1} are the coordinates for the subspace $\beta_1(y) = \beta_2(y) = 0$, and $\beta_1(y) = y_n$. It is easy to see that α_1 and α_2 must coincide on the subspace $y_n = 0$. Thus we have

$$\alpha_2^2 = \alpha_1^2 + (a_1y_1 + \dots + a_{n-1}y_{n-1})y_n + \mu y_n^2. \quad (3.2)$$

Taking the partial derivative with respect to y_n for the equation $\alpha_1\phi(\frac{\beta_1}{\alpha_1}) = \alpha_2\phi(\lambda\frac{\beta_1}{\alpha_2})$ and restricting y to the unit sphere $S = \{y | y_n = 0, \sum_{i=1}^{n-1} y_i^2 = 1\}$, we get

$$\phi(0)\frac{\partial}{\partial y_n}\alpha_2|_{y \in S} = \phi'(0)(1 - \lambda). \quad (3.3)$$

If $\lambda \neq 1$, then $\frac{\partial}{\partial y_n}\alpha_2|_{y \in S} = \frac{\partial}{\partial y_n}\alpha_2^2|_{y \in S}$ is a nonzero constant function. But by (3.2), it should also be a linear function $a_1y_1 + \dots + a_{n-1}y_{n-1}$, which must change signs. This is a contradiction. If $\lambda = 1$, then $\alpha_2^2 = \alpha_1^2 + \mu y_n^2$ for some $\mu > -1$. Denote $s = y_n/\alpha_1(y)$. Then the function ϕ satisfies

$$\phi(s) = \sqrt{1 + \mu s^2}\phi\left(\frac{s}{\sqrt{1 + \mu s^2}}\right). \quad (3.4)$$

Notice that s can take any value in an open neighborhood of 0. If $\mu \neq 0$, then the last equality implies that ϕ is neither increasing nor decreasing around 0. This conflicts the condition $\phi'(0) \neq 0$. Therefore $\mu = 0$ and α_1 and α_2 are identical.

Next we consider the case that β_1 and β_2 are linearly independent. Since $n \geq 3$, the two planes $\beta_1(y) = 0$ and $\beta_2(y) = 0$ have non-empty intersections on the ellipsoid $\alpha_1(y) = 1$. Hence there exists a sequence $\{y_n\}$ such that $\alpha_1(y_n) = 1$, $\beta_1(y_n) = 0$ and $\beta_2(y_n) \neq 0$, for any n , such that $\lim \beta_2(y_n) = 0$. Then we have $\phi(\frac{\beta_2(y_n)}{\alpha_2(y_n)}) = \phi(-\frac{\beta_2(y_n)}{\alpha_2(y_n)})$. The two sequences $\pm \left\{\frac{\beta_2(y)}{\alpha_2(y)}\right\}$ are both convergent to 0, contradicting the condition $\phi'(0) \neq 0$. This completes the proof of the lemma. ■

When the two pairs (α_i, β_i) , $i = 1, 2$ are connected by a smooth family (α_t, β_t) , such that they all define the same (α, β) -metric F , the condition $n > 2$ can be removed.

Lemma 3.3 Suppose the function ϕ satisfies $\phi'(0) \neq 0$ and there is a smooth family of pairs (α_t, β_t) which define the same metric $F = \alpha_t\phi(\beta_t/\alpha_t)$ on the Minkowski space \mathbb{R}^n , where $n \geq 2$. Then (α_t, β_t) is a constant family.

Proof. If $\beta_t = 0$, then $\alpha_t = \frac{F}{\phi(0)}$, which is independent of t . Hence We need only consider the the open intervals on which $\beta_t \neq 0$ for any t .

Suppose (a, b) is an interval such that $\beta_t \neq 0$ for any $t \in (a, b)$. Taking the partial derivative of $F = \alpha_t\phi(\frac{\beta_t}{\alpha_t})$ with respect to t , we have

$$\left(\phi\left(\frac{\beta_t}{\alpha_t}\right) - \frac{\beta_t}{\alpha_t}\phi'\left(\frac{\beta_t}{\alpha_t}\right)\right)\frac{\partial}{\partial t}\alpha_t + \phi'\left(\frac{\beta_t}{\alpha_t}\right)\frac{\partial}{\partial t}\beta_t = 0. \quad (3.5)$$

For any fixed t , both the functions $\phi(\frac{\beta_t}{\alpha_t}) - \frac{\beta_t}{\alpha_t}\phi'(\frac{\beta_t}{\alpha_t}) = \phi(0)$ and $\phi'(\frac{\beta_t}{\alpha_t}) = \phi'(0)$ are nonzero constant on the subspace $W = \{y | \beta_t(y) = 0\}$. The equality (3.5) implies that $\frac{\partial}{\partial t}\alpha_t$ and $\frac{\partial}{\partial t}\beta_t$ are linearly dependent functions on W . Given $w \in W$, we have

$$\frac{\partial}{\partial t}\alpha_t(cw) = |c|\frac{\partial}{\partial t}\alpha_t(w), \quad \frac{\partial}{\partial t}\beta_t(cw) = c\frac{\partial}{\partial t}\beta_t(w), \quad \forall c \in \mathbb{R}.$$

Since $\phi(0)$ and $\phi'(0)$ are both nonzero, the above equation shows that $\frac{\partial}{\partial t}\alpha_t(w)$ and $\frac{\partial}{\partial t}\beta_t(w)$ must be zero. In particular, we have a well-defined smooth function $f(t)$ such that $\frac{\partial}{\partial t}\beta_t = f(t)\beta_t$. Solving this simple partial differential equation, we see that all the β_t 's are linearly dependent to each other. By Lemma 3.2, the smooth family (α_t, β_t) are all identical. ■

Using the above two lemmas, we can give an explicit description of the isometries and Killing vector fields of a regular (α, β) -metric.

Theorem 3.4 *Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be a regular (α, β) -metric on M . Then a vector field X is a Killing vector field of F if and only if X is a Killing vector field of α and $L_X\beta = 0$.*

In particular, if $\dim M > 2$, then a diffeomorphism φ is an isometry of (M, F) if and only if $\varphi^\alpha = \alpha$ and $\varphi^*\beta = \beta$.*

Proof. We first prove the statement for isometries. The metric φ^*F is defined as

$$\varphi^*F(x, y) = \alpha(\varphi(x), \varphi_*y)\phi\left(\frac{\beta_{\varphi(x)}(\varphi_*y)}{\alpha(\varphi(x), \varphi_*y)}\right). \quad (3.6)$$

So it is also an (α, β) -metric, defined by the same function ϕ and the pair $(\varphi^*\alpha, \varphi^*\beta)$. By Lemma 3.2, if $\phi'(0) \neq 0$ and $\dim M > 2$, then $\varphi^*F = F$ if and only if $\varphi^*\alpha = \alpha$ and $\varphi^*\beta = \beta$. If φ lies in a one-parameter group of isometries generated by a vector field X , we can get corresponding conditions for X to be a Killing vector field by taking the differentiation of the parameter. In this case, F and φ^*F can be connected by a smooth family of (α, β) -metrics with the same ϕ . Hence by Lemma 3.3, the requirement on the dimension is not needed. ■

Now we can easily get a similar description of regular homogeneous (α, β) -metrics as in the Randers case.

Proposition 3.5 *Let (M, F) be a regular homogeneous (α, β) -space and G a connected closed transitive subgroup of $I_0(M, F)$ with $M = G/H$. Suppose either G is simply connected or $\dim M > 2$. Denote $\text{Lie}(G) = \mathfrak{g}$, $\text{Lie}(H) = \mathfrak{h}$, and $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$. Then the metric F is uniquely determined by an $\text{Ad}(H)$ -invariant inner product on \mathfrak{m} and an $\text{Ad}(H)$ -invariant vector in \mathfrak{m}^* .*

The $\text{Ad}(H)$ -invariant inner product and the $\text{Ad}(H)$ -invariant vector induce α and β , respectively, by the actions of G and the $\text{Ad}(H)$ -actions.

4 Killing vector fields of constant length

In this section, we consider a special case of regular homogeneous (α, β) -metric on a compact connected simple Lie group. A left invariant (α, β) -metric on a Lie group G

which is regular in the sense of Definition 3.1 will be called a regular left invariant (α, β) -metric on G .

Let G be a compact connected simple Lie group and $F = \alpha\phi(\frac{\beta}{\alpha})$ a regular left invariant (α, β) -metric on G . Then we have

Lemma 4.1 *The connected isometry group $I_0(G, F)$ of F is a subgroup of $L(G)R(G)$.*

Proof. Since $\dim G \geq 3$ and F is regular, Theorem 3.4 shows that $I_0(M, F)$ is a subgroup of $I_0(M, \alpha)$. Since the Riemannian metric α is also left invariant, $I_0(M, \alpha)$ is contained in $L(G)R(G)$ (see [OT76]). Thus $I_0(M, F) \subset L(G)R(G)$. ■

Remark. The lemma actually holds for any left invariant Finsler metric on G . In fact, one can define Riemannian metrics by taking average on the indicatrix of a Finsler metric F . In the literature, there are several ways to define the averaged Riemannian metric. Here we only present one of them. On each slit tangent space of a Finsler manifold, the Hessian matrix $g_{ij}(y) = \frac{1}{2}[F^2]_{y^i y^j}$ defines a Riemannian metric on the indicatrix (the induced metric), which depends only on F but not on the local coordinates. Denote the volume form on the indicatrix derived from this matrix as dv_y . Then F defines a Riemannian metric F' by

$$F'^2(y') = \int_{F=1} g_{ij}(y) y'^i y'^j dv_y, \quad (4.7)$$

where $y' = (y'^i)$. The metric F' is globally defined. Since any isometry of F must keep the indicatrix and the induced metric invariant, we have $I(M, F) \subset I(M, F')$. From this the assertion follows.

By Lemma 4.1, the group $I_0(G, F)$ is a product $L(G)R(G')$, where $R(G')$ is the maximal connected subgroup of isometric right translations. As the groups of left translations and right translations of G commute with each other, and their intersection is a finite subgroup, at the Lie algebra level, we have a direct sum decomposition $\text{Lie}(I_0(G, F)) = \mathfrak{g} \oplus \mathfrak{g}'$, where $\mathfrak{g} = \text{Lie}(G) = \text{Lie}(L(G))$ and $\mathfrak{g}' = \text{Lie}(G') = \text{Lie}(R(G'))$. This implies that any Killing vector field of (G, F) can be denoted as a pair $(X, X') \in \mathfrak{g} \oplus \mathfrak{g}'$.

If F is regular, then the inner product α and the vector β induced by F on \mathfrak{g} and \mathfrak{g}^* must be $\text{Ad}(G')$ -invariant. Denoting the α -dual of β by $V \in \mathfrak{g}$, we have $G' \subset C_G(V)$ and $\mathfrak{g}' \subset \mathfrak{c}_{\mathfrak{g}}(V)$.

The relationship between CW-translations and Killing vector fields reduces the problem of finding local one-parameter subgroups of CW-translations to the problem of finding Killing vector fields of constant length. For a regular (α, β) -metric F on a compact connected simple Lie group G , we have a similar criterion for a Killing vector field $(X, X') \in \mathfrak{g} \oplus \mathfrak{g}'$ to have constant length as in the Randers case [DM3].

Theorem 4.2 *If $(X, X') \in \mathfrak{g} \oplus \mathfrak{g}'$ defines a Killing field of constant length of a regular left invariant $F = \alpha\phi(\frac{\beta}{\alpha})$ on a compact connected simple Lie group G , then either $X = 0$ or $X' \in \mathfrak{c}(\mathfrak{g}')$.*

Proof. The Killing vector field generated by (X, X') has constant length if and only if $F(\text{Ad}(g)X - \text{Ad}(g')X')$ is a constant function of $g \in G$ and $g' \in G'$. In particular, for

any fixed $g \in G$, $\beta(\text{Ad}(g)X - \text{Ad}(g')X') = \beta(\text{Ad}(g)X) - \beta(X')$ is a constant function of g' . If $\beta(\text{Ad}(g)X - \text{Ad}(g')X') \neq 0$, then $\alpha(\text{Ad}(g)X - \text{Ad}(g')X')$ must also be a constant function of g' , otherwise it conflicts with the assumption that $\phi(s) - s\phi'(s) > 0$ for ϕ . If $\beta(\text{Ad}(g)X - \text{Ad}(g')X') = 0$ for all g' 's, then $\alpha(\text{Ad}(g)X - \text{Ad}(g')X')$ is a constant function of g' , since $F(\text{Ad}(g)X - \text{Ad}(g')X') = \phi(0)\alpha(\text{Ad}(g)X - \text{Ad}(g')X')$ is a constant. Thus for any $g \in G$, the vector $\text{Ad}(g)X$ is orthogonal to the ideal generated by $[X', \mathfrak{g}']$ in \mathfrak{g}' with respect to the metric α . Now varying g , we can see that the ideal of \mathfrak{g} generated by X is orthogonal to the ideal of \mathfrak{g}' generated by $[X', \mathfrak{g}']$ with respect to the metric α . Therefore either $X = 0$ or $X' \in \mathfrak{c}(\mathfrak{g}')$. ■

For simplicity, we denote the set of all Killing vector fields of constant length for the metric F as \mathcal{K}_F . Theorem 4.2 indicates that \mathcal{K}_F is the union of $\mathcal{K}_{1,F}$ and $\mathcal{K}_{2,F}$, where $\mathcal{K}_{1,F}$ is the closure of the set of all Killing vector fields (X, X') of constant length with $X \neq 0$, and $\mathcal{K}_{2,F}$ is the linear subspace $0 \oplus \mathfrak{g}'$. The next lemma asserts that the two subsets intersect only at 0.

Lemma 4.3 (1) *There is a constant $C > 0$, such that any Killing vector field of constant length (X, X') with $X \neq 0$ and $X' \in \mathfrak{c}(\mathfrak{g})$ satisfies $\frac{\|X'\|_\alpha}{\|X\|_\alpha} < C$.*

(2) $\mathcal{K}_{1,F} \cap \mathcal{K}_2 = 0$.

Proof. (1) The Lie algebra \mathfrak{g} will be viewed as a flat manifold with the metric $\langle \cdot, \cdot \rangle_{\text{bi}}$, and any submanifold in it will be equipped with the induced metric.

Suppose conversely that the constant $C > 0$ indicated by the lemma does not exist. Then there is a sequence of Killing vector fields (X_n, X'_n) , such that $\langle X_n, X_n \rangle_{\text{bi}} = 1$, and $X'_n \in \mathfrak{c}(\mathfrak{g}')$ with

$$\lim_{n \rightarrow \infty} F(\text{Ad}(g)X_n - X'_n) = \infty.$$

Denote $F(\text{Ad}(g)X_n - X'_n) = l_n$. By choosing a subsequence if necessary, we may assume that

$$\lim_{n \rightarrow \infty} X_n = X$$

Define the hypersurface $S_n = \{Y \in \mathfrak{g} | F(Y - X'_n) = l_n\}$, which is a parallel shifting of the hypersurface $F = l_n$ with the origin moved to $-X'_n$. The hypersurface S_n in \mathfrak{g} contains the $\text{Ad}(G)$ -orbit of X_n and Their principal curvatures uniformly converge to 0. Within the unit ball centered at 0, the distance between a point X on S_n with its orthogonal projection $\text{pr}_n(X)$ to the tangent space L_n of S_n at some point in the unit ball is an $o(1)$ quantity when n goes to infinity. By choosing a subsequence if necessary, we can assume that L_n converge to a hyperplane L of \mathfrak{g} . Then the sequence $\text{pr}_n(X)$ converges to the orthogonal projection $\text{pr}(X)$ to L . Therefore for any $g \in G$, we have

$$\begin{aligned} \text{Ad}(g)X &= \lim_{n \rightarrow \infty} \text{Ad}(g)X_n \\ &= \lim_{n \rightarrow \infty} \text{pr}_n(\text{Ad}(g)X_n) + \lim_{n \rightarrow \infty} (\text{Ad}(g)X_n - \text{pr}_n(\text{Ad}(g)X_n)) \\ &= \text{pr}(\text{Ad}(g)X). \end{aligned} \tag{4.8}$$

This indicates that the $\text{Ad}(G)$ -orbit of the unit vector X is contained in L , contradicting the assumption that \mathfrak{g} is simple.

(2) We only need to consider those pairs (X, X') which is the limit of the sequence (X_n, X'_n) in $\mathcal{K}_{1,F}$, i.e., $X_n \neq 0$. If

$$X = \lim_{n \rightarrow \infty} 0,$$

then from (1), we have

$$X' = \lim_{n \rightarrow \infty} 0.$$

Thus in $\mathcal{K}_{2,F}$, only the origin 0 is contained in the derived set of $\mathcal{K}_{1,F}$. Therefore we have $\mathcal{K}_{1,F} \cap \mathcal{K}_2 = 0$. ■

This lemma is useful for understanding the structure of Killing vector fields of constant length and the corresponding CW-translations, especially when F is CW-homogeneous. We will discuss the details in the next section.

Here is another lemma needed in the next section, which implies that for any fixed X , the set

$$\{X' \mid (X, X') \in \mathcal{K}_{1,F}\}$$

should be small.

Theorem 4.4 *Let the notations be as above. Suppose (X, X') and (X, X'') are contained in $\mathcal{K}_{1,F}$. Then there exists $c \in \mathbb{R}$ such that $X' - X'' = cV$, where V is the α -dual of β .*

Remark. In the special case that F is Riemannian, i.e., $\beta = 0$, for each nonzero $X \in \mathfrak{g}$, there is at most one X' such that $(X, X') \in \mathcal{K}_{F,1}$.

Proof. Lemma 4.3 indicates that, when $X = 0$, we have $X' = X'' = 0$. So we can assume that $X \neq 0$. The condition that $(X, X') \in \mathcal{K}_{1,F}$ implies that

$$F(\text{Ad}(g_t g)X - X') = \alpha(\text{Ad}(g_t g)X - X') \phi\left(\frac{\beta(\text{Ad}(g_t g)X - X')}{\|\text{Ad}(g_t g)X - X'\|_\alpha}\right) = \text{const}, \quad (4.9)$$

where $g_t = \exp(tY)$. Differentiating the equation with respect to t , and letting $t = 0$, we get

$$\frac{\phi(s) - s\phi'(s)}{\alpha(\text{Ad}(g)X - X')} \langle [Y, \text{Ad}(g)X], \text{Ad}(g)X - X' \rangle_\alpha + \phi'(s) \langle [Y, \text{Ad}(g)X], V \rangle_\alpha = 0, \quad (4.10)$$

where $s = \beta(\text{Ad}(g)X - X')/\alpha(\text{Ad}(g)X - X')$. A similar equality holds for (X, X'') . Notice that the coefficient $\frac{\phi(s) - s\phi'(s)}{\alpha(\text{Ad}(g)X - X')}$ must be nonzero. Now given $g \in G$ and $Y \in \mathfrak{g}$, if $\langle [Y, \text{Ad}(g)X], V \rangle_\alpha = 0$, then

$$\langle [Y, \text{Ad}(g)X], \text{Ad}(g)X - X' \rangle_\alpha = \langle [Y, \text{Ad}(g)X], \text{Ad}(g)X - X'' \rangle_\alpha = 0. \quad (4.11)$$

Therefore $\langle [Y, \text{Ad}(g)X], X' - X'' \rangle_\alpha = 0$.

If F is Riemannian, i.e., if $\beta = 0$ (or equivalently, $V = 0$), then $X' - X''$ is α -orthogonal to all the tangent spaces of the $\text{Ad}(g)$ -orbit \mathcal{O}_X . In this case we have $X' - X'' = 0$.

Now assume $V \neq 0$. Let U be the dual of β with respect to the bi-invariant metric. We prove that the set

$$S = \{Y \mid Y = [Y', \text{Ad}(g)X] \text{ for some } g \in G, Y' \in \mathfrak{g} \text{ and } \langle Y, V \rangle_\alpha = 0\} \quad (4.12)$$

spans the subspace $\langle Y, V \rangle_\alpha = 0$. This will complete the proof of the theorem. It suffices to prove that \mathcal{S} spans U^\perp . If there is a nonzero $U' \in \mathcal{S}^{\perp_{bi}}$ linearly independent with U , then the subspace L spanned by U and U' has a dimension $\dim L = 2$. Hence for any $g \in G$, the intersection of L with $[\mathfrak{g}, \text{Ad}(g)X]^{\perp_{bi}} = \text{Ad}(g)\mathfrak{c}_{\mathfrak{g}}(X)$ contains nonzero vectors. But the next lemma shows that this can not be true. ■

Lemma 4.5 *Let G be a compact connected simple Lie group with Lie algebra \mathfrak{g} , $X \in \mathfrak{g}$ be a nonzero vector, and $L \subset \mathfrak{g}$ be a subspace with $\dim L = 2$. Then there exists $g \in G$, such that $L \cap \text{Ad}(g)\mathfrak{c}_{\mathfrak{g}}(X) = \{0\}$.*

Proof. Assume conversely that for any $g \in G$, $L \cap \text{Ad}(g)\mathfrak{c}_{\mathfrak{g}}(X)$ contains nonzero vectors. The subset of all g 's in G such that $\dim L \cap \text{Ad}(g)\mathfrak{c}_{\mathfrak{g}}(X) = 2$ is closed in G , and it can not be G itself, otherwise L is contained in the ideal $\cap_{g \in G} \text{Ad}(g)\mathfrak{c}_{\mathfrak{g}}(X)$, which must be 0. Suitably changing X within its conjugation class, we can assume $L \cap \mathfrak{c}_{\mathfrak{g}}(X)$ is one dimensional and generated by U . Let V be another vector such that L is generated by V . Then for any $g \in G$ close to e , $L \cap \text{Ad}(g)\mathfrak{c}_{\mathfrak{g}}(X)$ is one dimensional and generated by $U + f(g)V$, where $f(g)$ is a smooth function, satisfying $f(0) = 0$ and $[U + f(g)V, \text{Ad}(g)X] = 0$. Differentiating the above equation at $g = e$, we get $[Df(Y)V + [Y, U], X] = 0$, $\forall Y \in \mathfrak{g}$, i.e., $[\mathfrak{g}, U] \subset \mathbb{R}V + \mathfrak{c}_{\mathfrak{g}}(X)$. So $\dim[\mathfrak{g}, U] - \dim[\mathfrak{g}, U] \cap \mathfrak{c}_{\mathfrak{g}}(X) \leq 1$. Suitably changing X with a Weyl group action, we can further assume that X and U belong to the same closed Weyl chamber in a Cartan subalgebra. Since U and X are nonzero vectors, the highest root provides two dimensions in $[\mathfrak{g}, U]$ which is not in $[\mathfrak{g}, U] \cap \mathfrak{c}_{\mathfrak{g}}(X)$. This indicates that $\dim[\mathfrak{g}, U] - \dim[\mathfrak{g}, U] \cap \mathfrak{c}_{\mathfrak{g}}(X) \geq 2$, which is a contradiction. ■

5 Proof of Theorem 1.1

Now we give the proof of Theorem 1.1. We will keep the notations of the last section. In particular, G is a compact connected simple Lie group, and $F = \alpha\phi(\frac{\beta}{\alpha})$ is a regular left invariant (α, β) -metric on G . The connected isometry group is $I_0(G, F) = L(G)R(G')$, where $R(G')$ is the maximal connected subgroup of isometric right translations. The left invariant metric α induces an $\text{Ad}(G')$ -invariant inner product $\|\cdot\|_\alpha^2 = \langle \cdot, \cdot \rangle_\alpha$ on \mathfrak{g} . We also fix a bi-invariant inner product $\|\cdot\|_{bi}^2 = \langle \cdot, \cdot \rangle_{bi}$. The β -term of F can then be expressed as $\langle \cdot, V \rangle_\alpha = \langle \cdot, U \rangle_{bi}$, where V and U are two fixed vectors in \mathfrak{g} . Note that both V and U are $\text{Ad}(G')$ -invariant.

Lemma 5.1 *Let F be a regular left invariant (α, β) -metric on a connected compact simple Lie group G . If F is non-Riemannian and CW-homogeneous, then*

- (1) *The function ϕ is analytic on $[-\epsilon_0, \epsilon_0]$, where $\epsilon_0 = \sup_{Y \in \mathfrak{g} \setminus 0} \frac{\langle Y, V \rangle_\alpha}{\|Y\|_\alpha}$.*
- (2) *The set $\mathcal{K}_F \setminus 0$ is a closed real analytic subvariety in $\mathfrak{g} \oplus \mathfrak{g}' \setminus 0$.*
- (3) *For each $X \in \mathfrak{g}$, there are at most finitely many X' , which differ by multiples of V , such that $(X, X') \in \mathcal{K}_{1,F}$.*

Proof. (1) Let $(X, X') \in \mathfrak{g} \oplus c(\mathfrak{g}')$ be a Killing vector field of constant length with $X \neq 0$. Then around each $g_0 \in G$, the function $\frac{\langle \text{Ad}(g)X - X', V \rangle_\alpha}{\|\text{Ad}(g)X - X'\|_\alpha}$ can not be a constant function of g . In fact, otherwise we have $F(\text{Ad}(g)X - X') = \text{const}$. Then $\langle \text{Ad}(g)X, V \rangle_\alpha$ is a constant function, hence V is orthogonal to the ideal generated by $[\text{Ad}(g_0)X, \mathfrak{g}]$, i.e., $V = 0$. This is a contradiction with the assumption that F is non-Riemannian. This proves the assertion. Let $Y \in \mathfrak{g}$ be such that

$$f(t) = \frac{\langle \text{Ad}(\exp(tY))\text{Ad}(g)X - X', V \rangle_\alpha}{\|\text{Ad}(\exp(tY))\text{Ad}(g)X - X'\|_\alpha}$$

is not constant around $t = 0$. The function $f(t)$ is real analytic at $t = 0$. Let $f^{(n)}(0)$ be the first nonzero derivative of f at 0 with $n > 0$. Then with a real analytic change of variable from t to $t' = t(|f(t) - f(0)|/t^n)^{1/n}$, we have $f(t) = f(0) \pm t'^n$. The condition that (X, X') is a Killing vector field of constant length gives

$$\phi\left(\frac{\langle \text{Ad}(\exp(tY))\text{Ad}(g)X - X', V \rangle_\alpha}{\|\text{Ad}(\exp(tY))\text{Ad}(g)X - X'\|_\alpha}\right) = \frac{\text{const}}{\alpha(\text{Ad}(\exp(tY))\text{Ad}(g)X - X')}. \quad (5.13)$$

The left side of (5.13) is $\phi(f(t)) = \phi(f(0) \pm t'^n)$, with non-zero derivatives at $t' = 0$ only for the degrees which are multiples of n . So the same assertion holds for the right side of (5.13). Obviously the right side of (5.13) is an analytic function around $t' = 0$, so it is an analytic function of $f(t) = f(0) \pm t'^n$ at least at one side of $f(0)$. If $f(0)$ is neither a maximum nor a minimum of $\langle \text{Ad}(g)X - X' \rangle_\alpha / \|\text{Ad}(g)X - X'\|_\alpha$ for $g \in G$, we can change Y so that the corresponding $f(t)$ approaches the same $f(0)$ from another side. Then the smoothness of ϕ implies that it is real analytic around this point. Next we consider the case that $f(0)$ is a maximum or a minimum. In this case we need to change the Killing vector fields (X, X') .

By Theorem 2.5, the Killing vector fields (X, X') of constant length for F with $X \neq 0$ must exhaust all directions in $\mathfrak{g} = TG_e$ outside the subspace \mathfrak{g}' . Obviously the $\frac{\beta}{\alpha}$ -values of these directions cover the open interval $(-\epsilon_0, \epsilon_0)$. By the above argument, at each $s_0 \in (-\epsilon_0, \epsilon_0)$, ϕ is at least analytic at one side of s_0 , i.e., ϕ can be expanded as a power series of $s - s_0$ for $s \in [s_0, s_0 + \epsilon)$ or $s \in (s_0 - \epsilon, s_0]$ for some $\epsilon > 0$. For the other side s_0 , we can either find another Killing vector field with its $\frac{\beta}{\alpha}$ -values cover s_0 from the other side, or find a sequence of Killing vector fields (X_n, X'_n) of constant length 1 (for constant scalar changes do not affect the discussion), such that $X_n \neq 0$ for each n , and their $\frac{\beta}{\alpha}$ -images converge to an interval which cover s from the other side. In the second case, applying Lemma 4.3 we can find a subsequence of (X_n, X'_n) which converge to a Killing vector field of constant length 1 in $\mathcal{K}_{1,F}$, whose $\frac{\beta}{\alpha}$ -values cover s from the other side. So it reduces to the first case, which has been discussed. Because ϕ is smooth and analytic from both sides of s_0 , it is analytic at that point. For $s = \pm\epsilon_0$, we only need ϕ to be analytic from the side which is relevant to the metric F , the proof is similar and simpler.

(2) When ϕ is real analytic, the condition for a nonzero Killing vector field (X, X') to be contained in $\mathcal{K}_{1,F}$ can be expressed as a system of real analytic equations, namely, $F(\text{Ad}(g)X - X') = F(X - X')$, $\forall g \in G$. So $\mathcal{K}_{1,F} \setminus 0$ is a closed real analytic subvariety of $\mathfrak{g} \oplus \mathfrak{g}'$. The same assertion is obviously true for $\mathcal{K}_{2,F} \setminus 0$, and for their union $\mathcal{K}_F \setminus 0$.

(3) For $X = 0$, the assertion follows directly from Lemma 4.3. Now suppose $X \neq 0$, and there are infinitely many X' 's such that (X, X') is a Killing vector field of constant length. By Lemma 4.3, the set of those X' 's must be bounded. By Theorem 4.4, those X' 's differ by multiples of V . Then there is a sequence of real numbers t_n with limit 0, and a Killing vector field of constant length (X, X') , such that $F(\text{Ad}(g)X - X' + t_n V) \equiv C_n, \forall g \in G$. Since ϕ is real analytic, the function $F(\text{Ad}(g)X - X' + tV)$ is a continuous function of g and t , and it is real analytic whenever $\text{Ad}(g)X - X' + tV \neq 0$. Since $X \neq 0$, the set of $g \in G$ such that $\text{Ad}(g)X - X' + tV = 0$ for some t is a union of finite conjugate classes of $C_G(X)$, which has a codimension at least 2 in G . Then its complement in G is dense open and connected. Now given g such that $\text{Ad}(g)X - X' + tV \neq 0$ for all t , $F(\text{Ad}(g)X - X' + tV)$ is a real analytic function of t . The above real analytic function are the same for all g , Since they all coincide for a converging sequence t_n . By the continuity, $F(\text{Ad}(g)X - X' + tV)$ does not depend on g , i.e., $(X, X' - tV) \in \mathcal{K}_{1,F}$ for all t , which is a contradiction with Lemma 4.3. ■

There are two natural projections from $\mathcal{K}_{1,F}$ to \mathfrak{g} , namely,

$$\pi_1(X, X') = X - X', \quad \text{and} \quad \pi_2(X, X') = X. \quad (5.14)$$

It can be easily seen from Lemma 4.3 that, when (G, F) is CW-homogeneous, the map π_1 is surjective. Thus the dimension of the real analytic variety $\mathcal{K}_{1,F}$ is no less than $\dim \mathfrak{g}$. Lemma 5.1 indicates that π_2 has a finite pre-image for each X , which implies that the dimension of $\mathcal{K}_{1,F}$ must be exactly $\dim \mathfrak{g}$, and the image of π_2 must contain an open set of \mathfrak{g} .

More precisely, the real analytic variety $\mathcal{K}_{1,F} \setminus 0$ can be locally stratified as a finite union of smooth manifolds. Let \mathcal{U} be a small open set in a stratum with the top dimension (i.e., the dimension of $\mathcal{K}_{1,F} \setminus 0$). The projection π_2 restricted to \mathcal{U} must be regular at some point of \mathcal{U} . Otherwise we have $l = \min \dim \ker \pi_{2*} > 0$. There is smaller open set \mathcal{U}' on which $\ker \pi_{2*}$ is l -dimensional smooth distribution. Then along any flow curve for a nonzero smooth section of $\ker(\pi_2)_*$ in \mathcal{U}' , π_2 is constant. This is a contradiction with Lemma 5.1. So \mathcal{U} has the same dimension as \mathfrak{g} , and around the regular point of \mathcal{U} , π_2 is a diffeomorphism onto an open set of \mathfrak{g} .

Let \mathfrak{t} be a Cartan subalgebra containing $\mathfrak{c}(\mathfrak{g}')$ (which also contains V). Up to suitable conjugations, we can assume that $\text{Im}(\pi_2)$ contains a non-empty open set $\mathcal{U}'' \subset \mathfrak{t}$. For any nonzero $X \in \mathcal{U}''$ and $g \in G$, (4.10) implies that there is a constant c depending on X and g , and a $X' \in \mathfrak{t}$ which does not depend on g and α -orthogonal to V , such that

$$\langle Y, \text{Ad}(g)X - X' + cV \rangle_\alpha = 0, \quad \forall Y \in [\text{Ad}(g)X, \mathfrak{g}]. \quad (5.15)$$

Lemma 4.4 indicates that, whenever there is a solution pair (X', c) for $X \neq 0$, X' is uniquely determined by X . Given regular vectors $X_i \in \mathcal{U}''$, $i = 1, \dots, n$. Denote the corresponding solution pairs as (X'_i, c_i) , $i = 1, \dots, n$ (the c_i 's may not be unique). For any $g \in G$, the linear subspaces $[\text{Ad}(g)X_i, \mathfrak{g}] = \text{Ad}(g)[X_i, \mathfrak{g}]$ are the same for all $i = 1, \dots, n$. Then for any linear combinations $X = \sum_{i=1}^n a_i X_i$, $X' = \sum_{i=1}^n a_i X'_i$, and $c = \sum_{i=1}^n a_i c_i$, we have

$$\langle Y, \text{Ad}(g)X - X' + cV \rangle_\alpha = 0, \quad (5.16)$$

for all $Y \in [\text{Ad}(g)X, \mathfrak{g}] \subset [\text{Ad}(g)X_i, \mathfrak{g}]$ for each i , i.e., (X', c') is a solution pair of (5.15) for $X = \sum_{i=1}^n a_i X_i$. So there is a well-defined linear map from X to X' which solves

(5.16). Since it is invariant under the Weyl group action, this map must be the zero endomorphism. To summarize, we have the following lemma.

Lemma 5.2 *Let (G, F) be a non-Riemannian CW-homogeneous regular (α, β) -space, with $F = \alpha\phi(\frac{\beta}{\alpha})$, and $\beta = \langle \cdot, V \rangle_\alpha$. Then for any $X \in \mathfrak{g}$, there is $c \in \mathbb{R}$ such that*

$$\langle Y, X \rangle_\alpha = \langle Y, cV \rangle_\alpha = 0, \forall Y \in [X, \mathfrak{g}]. \quad (5.17)$$

Lemma 5.2 implies an important necessary condition for the (α, β) -metric to be CW-homogeneous, namely, the Riemannian metric α must be very close to the bi-invariant one. To be precise, we have the following proposition.

Proposition 5.3 *Let (G, F) be a non-Riemannian CW-homogeneous regular (α, β) -space, with $F = \alpha\phi(\frac{\beta}{\alpha})$. Then there are constants a and b , such that $\|\cdot\|_\alpha^2 = a\|\cdot\|_{bi}^2 + b\beta^2(\cdot)$.*

Proof. If $[X, \mathfrak{g}]$ is not contained in $\ker \beta$, then the constant c in Lemma 5.2 is uniquely determined by X . Otherwise (5.17) is satisfied for all $c \in \mathbb{R}$. If we write β as $\langle \cdot, U \rangle$, the condition $[X, \mathfrak{g}] \subset \ker \beta$ is just $[U, X] = 0$. So it is unambiguous to denote $c = c(X)$, when $[X, U] \neq 0$.

If α is not bi-invariant, then by (5.17), for any fixed $Z \in \mathfrak{g}$ such that $[Z, U] \neq 0$, or equivalently $\langle [Z, X], V \rangle_\alpha \neq 0$ for some X , the bi-linear function $f(X) = \langle [Z, X], X \rangle_\alpha$ vanishes on the codimension one linear subspace $\langle [Z, X], V \rangle_\alpha = 0$. This can only happen when $f(X)$ splits as the product of two linear factors. One of the two linear factors is $\langle [Z, X], V \rangle_\alpha$, and the other must coincide with $c(X)$ when $\langle [Z, X], V \rangle_\alpha \neq 0$. Whenever $[U, X] \neq 0$ (and $c(X)$ is well-defined), there is a $Z \in \mathfrak{g}$, such that $\langle [Z, X], V \rangle_\alpha \neq 0$. So for any Z with $[Z, U] \neq 0$, the quotient $f(X)/\langle [Z, X], V \rangle_\alpha$ defines the same linear function which can be identified with $c(X)$ for $[U, X] \neq 0$. So we have a linear function $c(X)$ on \mathfrak{g} such that

$$\langle Y, X \rangle_\alpha = c(X) \langle Y, V \rangle_\alpha, \quad \forall Y \in [X, \mathfrak{g}]. \quad (5.18)$$

The bilinear function $L(X, Y) = \langle Y, X \rangle_\alpha - c(X) \langle Y, V \rangle_\alpha$ on $\mathfrak{g} \times \mathfrak{g}$ vanishes whenever $Y \in [X, \mathfrak{g}]$, so it vanishes on $\mathfrak{h} \times \mathfrak{h}^\perp$, for any Cartan subalgebra \mathfrak{h} and its orthogonal complement \mathfrak{h}^\perp with respect to $\langle \cdot, \cdot \rangle_{bi}$. The function $L(X, Y)$ defines a linear map $l : \mathfrak{g} \rightarrow \mathfrak{g}$, such that $L(X, Y) = \langle l(X), Y \rangle_{bi}$. Then l maps each Cartan subalgebra to itself. There is a nonzero vector $X \in \mathfrak{g}$ such that $\mathbb{R}X$ is the intersection of a set of Cartan subalgebras. Then any vector in the $\text{Ad}(G)$ -orbit of X is a eigenvector of l . This can hold only when they belong to the same eigenvalue. Since G is simple, the vectors in orbit of X generate the whole \mathfrak{g} . Thus l is a multiple of the identity map, and $L(X, Y)$ is a bi-invariant inner product on \mathfrak{g} . Therefore $c(X)$ is a multiple of β . ■

We are now ready to prove the main theorem.

Proof of Theorem 1.1. First consider the case that there is a nonzero $X \in \mathfrak{g}$, such that $(X, 0) \in \mathcal{K}_{1,F}$. We can change the presentation of F with α changed to a bi-invariant metric, β unchanged, and the new function ϕ still satisfying the regular property $\phi'(0) \neq 0$. More precisely, if $\alpha^2 = a\tilde{\alpha}^2 + b\beta^2$, where $\tilde{\alpha}$ is bi-invariant, $\tilde{\beta} = \beta$,

and $\tilde{\phi}(s) = \sqrt{a + bs^2}\phi(\frac{s}{\sqrt{a+bs^2}})$, then we have $F = \tilde{\alpha}\tilde{\phi}(\tilde{\beta}/\tilde{\alpha})$, and $\tilde{\phi}'(0) \neq 0$. Since $(X, 0) \in \mathcal{K}_{1,F}$, we have

$$F(\text{Ad}(g)X) = \tilde{\alpha}(\text{Ad}(g)X)\tilde{\phi}(\frac{\tilde{\beta}(\text{Ad}(g)X)}{\tilde{\alpha}(\text{Ad}(g)X)}) = \text{const.} \quad (5.19)$$

Since $\tilde{\alpha}(\text{Ad}(g)X) = \tilde{\alpha}(X)$ is a nonzero constant function of $g \in G$, and $\tilde{\beta}(\text{Ad}(g)X)$ is not a constant, the real analytic function $\tilde{\phi}$ must be constant, i.e., F must be a bi-invariant Riemannian metric. Note that this cannot happen when F is regular. Note also that the above argument is still valid without the regular assumption on F . This observation will be useful in the following.

In the case that there does not exist a nonzero $X \in \mathfrak{g}$ such that $(X, 0) \in \mathcal{K}_{1,F}$, one can find a Killing vector field $(X, \lambda V) \in \mathcal{K}_{1,F}$ of constant length 1, with $\lambda \neq 0$ and $X \neq 0$. The strong convexity of F indicates that $F(-\lambda V) < 1$. Applying a navigation transformation to F which set the new origin at $-\lambda V$, we get a new metric \tilde{F} . The indicatrix of \tilde{F} is a parallel shift of that of F , with $-\lambda V$ shifted to 0. Suppose σ is an isometry of F . Then σ preserves both the indicatrix of F and the vector field V . Hence its action on the tangent bundle commutes with the parallel shift by $-\lambda V$. Thus σ is also an isometry of \tilde{F} . But then $(X, 0)$ is a Killing field of \tilde{F} . The geometry property of navigation transformation indicates that $(X, 0)$ is of constant length 1. Writing $\|\cdot\|_{\alpha}^2 = a\|\cdot\|_{\text{bi}}^2 + b\beta^2(\cdot)$, with $a, b \in \mathbb{R}$, one easily sees that the new metric \tilde{F} is still an (α, β) -metric, and its α -term is also of the above form. The argument in the previous paragraph then implies that \tilde{F} is a bi-invariant Riemannian metric. Therefore F must be a CW-homogeneous Randers space, completing the proof of the theorem.

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